ADIABATIC INVARIANTS AND THE "THIRD" INTEGRAL

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ABSTRACT

In a general case of Hamiltonian systems of n degrees of freedom, depending periodically on the time, n formal "third" integrals of motion are found. Their application in finding boundaries for the orbits is illustrated in a special case. Then a comparison is made between these integrals and the adiabatic invariants. Both are series expansions but the small parameter used is of different character in each case. This is shown explicitly in a simple example and the relative accuracy of the two expansions is discussed.

I. INTRODUCTION

It is well known that, under certain conditions, adiabatic invariants are constant to all orders in the small parameter[1], i.e., they are formal integrals of motion.

On the other hand in most time independent Hamiltonian systems of n degrees of freedom one can find n formal integrals of motion[2], as power series in the coordinates and momenta. We call any such integral a "third" integral, to distinguish it from the classical energy and angular momentum integrals.

The same method can be applied to a general case of n degrees of freedom, when the potential is expressed as a series in the coordinates, and is periodic in time. Then n formal integrals are found, which are periodic in time.

There are a few similarities between these integrals and the adiabatic invariants, but some important differences also. These are illustrated in the last section, where both a "third" integral and an adiabatic invariant are constructed for the same dynamical system.

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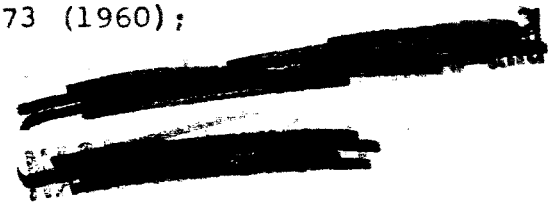
[1] See M. Kruskal, J. Math. Phys., 3, 806(1962) and references there.

[2] G. Contopoulos, Z. Astrophysik, 49, 273 (1960); Astron. J. 68, 1 (1963).



II. INTEGRALS OF MOTION IN PERIODIC POTENTIALS

Suppose that a potential is given as a series in the coordinates, beginning with terms of second degree, and periodic with respect to the time, with period $2\pi/\omega$. Then the Hamiltonian

$$H = H_2 + H_3 + \dots , \tag{1}$$

is also periodic in t, and it has the origin as equilibrium point. We will consider the case when the characteristic exponents of the equilibrium solution are pure imaginary and not equal. This is a most common case in applications. Then we can use a linear transformation of the variables , with coefficients periodic in t, and find a new Hamiltonian of the same form, where

$$H_2 = \sum_{i=1}^{n} \frac{\omega_i}{2} (x_i'^2 + y_i'^2) \tag{2}$$

in the new variables[3]. If we introduce further new coordinates and momenta,

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[3] A. Liapounoff, Problème Général de la Stabilité du Mouvement, Ann. Fac. Sci. Toulouse, 2nd Ser. 9 (1907), p. 281, 398.

$$x_i = x_i'/\sqrt{\omega_i} \ , \qquad y_i = \sqrt{\omega_i}\, y_i' \ , \tag{3}$$

we find

$$H_2 = \frac{1}{2} \sum_{i=1}^{n} (\omega_i^2 x_i^2 + y_i^2), \tag{4}$$

i.e., the second order Hamiltonian is independent of time and represents a system of n harmonic oscillators. Then H_3, H_4 ... are homogeneous polynomials in x_i, y_i, of degree 3, 4 ..., periodic in t, with period $2\pi/\omega$; hence the coefficients of the different terms can be given in the form $\sum c \begin{matrix}\sin \\ \cos\end{matrix} m\omega t \prod x_i^{a_i} y_i^{b_i}$ with integer m, a_i, b_i, and c constant.

Let us assume further that no relation of the form

$$m_1 \omega_1 + m_2 \omega_2 + \ldots + m\omega = 0 \tag{5}$$

exists with integer m_1, m_2, and m equal to any of the above given values. If m takes also the value 0, we assume that (5) is not satisfied, unless $m_1 = m_2 = \ldots = 0$.

Then we can construct, step by step, n integrals in the form of series

$$\Phi_i = \Phi_{i2} + \Phi_{i3} + \Phi_{i4} + \dots , \tag{6}$$

where

$$\Phi_{i2} = \frac{1}{2}(\omega_i^2 x_i^2 + y_i^2) , \tag{7}$$

and $\Phi_{i\nu}$ is a homogeneous polynomial of degree ν in x_i, y_i, periodic in t, with period $2\pi/\omega$.

In fact any integral (6) satisfies the equation

$$\frac{\partial \Phi_i}{\partial t} + (\Phi_i, H) \equiv \frac{\partial \Phi_i}{\partial t} + \sum_{i=1}^{n} \left(\frac{\partial \Phi_i}{\partial x_i} \frac{\partial H}{\partial y_i} - \frac{\partial \Phi_i}{\partial y_i} \frac{\partial H}{\partial x_i} \right) = 0 , \tag{8}$$

which can be split into the equations

$$\frac{\partial \Phi_{i2}}{\partial t} + (\Phi_{i2}, H_2) = 0 , \tag{9}$$

.....

$$\frac{\partial \Phi_{i,\nu+1}}{\partial t} + (\Phi_{i,\nu+1}, H_2) + (\Phi_{i,\nu}, H_3) + (\Phi_{i,\nu-1}, H_4) + \dots = 0 . \tag{10}$$

Equation (10) is a linear partial differential equation that gives $\Phi_{i,\nu+1}$ when the previous terms of the series Φ_i are known. The corresponding system to this equation is

$$dt = \frac{dx_i}{y_i} = \frac{dy_i}{-\omega_i^2 x_i} = \frac{d\phi_{i,\nu+1}}{K_{i,\nu}} \quad , \tag{11}$$

where the function

$$K_{i,\nu} = -(\phi_{i,\nu}, H_3) - (\phi_{i,\nu-1}, H_4) - \ldots \tag{12}$$

is known, and is of degree $\nu+1$.

This gives

$$\begin{aligned} x_i &= (\sqrt{2\phi_{i2}}/\omega_i)\sin \omega_i(t-t_i), \\ y_i &= \sqrt{2\phi_{i2}} \cos \omega_i(t-t_i) . \end{aligned} \tag{13}$$

Then

$$\phi_{i,\nu+1} = \int K_{i,\nu}\, dt \quad , \tag{14}$$

where $K_{i,\nu}$ is written in the form

$$\sum q(2\phi_{1_2})^{\frac{|m_1|}{2}+n_1}(2\phi_{2_2})^{\frac{|m_2|}{2}+n_2} \cdots \begin{matrix}\sin \\ \cos\end{matrix}\left(m_1\omega_1(t-t_1) + m_2\omega_2(t-t_2)+\ldots+ m\omega t\right), \tag{15}$$

where m_1, m_2, n_1, n_2 are integers, $n_1 > 0$, $n_2 > 0$, $\frac{1}{2}(|m_1|+|m_2|+\ldots) + n_1 + n_2 +\ldots = \nu+1$, and q are constants. If no coefficient (5) is zero, then equation (14) is integrated and gives $\phi_{i,\nu+1}$ as a sum of the same form as (15). This can be expressed as a polynomial of degree $\nu+1$ in x_i, y_i, with coefficients of the form $\begin{matrix}\sin \\ \cos\end{matrix} m\omega t$.

If H contains time independent terms, the corresponding m is zero. Then if $K_{i,\nu}$ contains a term with a cosine and $m_1 = m_2 = \ldots = 0$, this will give a secular term in $\xi_{i,\nu+1}$. It can be proved, however, as in the two dimensional case[2], that $K_{i,\nu}$ never includes cosines with $m_1=m_2=\ldots=0$. Therefore n formal integrals can always be constructed, step by step.

If for certain values of $m_1, m_2, \ldots m$, the corresponding quantity (5) is zero, the above integrals are no more valid. Then, however, the system (11) has further the integrals

$$\begin{matrix} S_M \\ C_M \end{matrix} = (2\xi_{1_2})^{\frac{|m_1|}{2}} (2\xi_{2_2})^{\frac{|m_2|}{2}} \ldots \begin{matrix} \sin \\ \cos \end{matrix} \left(m_1\omega_1(t-t_1) + m_2\omega_2(t-t_2) + \ldots + m_i\omega t\right), \tag{16}$$

which are polynomials in x_i, y_i of degree $M = |m_1| + |m_2| + \ldots$. Then one can construct two integrals of the form

$$\begin{aligned} S &= S_M + S_{M+1} + \ldots , \\ C &= C_M + C_{M+1} + \ldots , \end{aligned} \tag{17}$$

which will have also secular terms.

In a simple non-linear case it has been proved[4] that a combination of these integrals with the above integrals ξ_i can eliminate the secular terms and give n formal time independent integrals of motion. These resonance integrals

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[4] G. Contopoulos, Astron. J. 68, 763 (1963).

may be rather different from the above integrals (6). It seems probable that one can find such "resonant" integrals in the case of time-dependent potentials of the form (1) also.

In many problems the Hamiltonian is given in the form

$$H = \frac{1}{2}\sum_{i=1}^{n}(\alpha_i^2 x_i^2 + y_i^2) + \epsilon H_\epsilon , \tag{18}$$

where ϵ is a small parameter and H_ϵ is of degree > 2. Then we find integrals of the form

$$\Phi_i = \bar{\Phi}_{i(0)} + \epsilon \bar{\Phi}_{i(1)} + \epsilon^2 \bar{\Phi}_{i(2)} + \ldots, \tag{19}$$

where $\bar{\Phi}_{i(0)} = \Phi_{i2}$, and

$$\bar{\Phi}_{i,\ (\nu+1)} = -\int (\bar{\Phi}_{i,\ (\nu)}, H_\epsilon)dt . \tag{20}$$

The integrals Φ_i are useful in giving bounds for the orbits. E.g., in a two-dimensional time independent potential the boundary $f(x_1, x_2) = 0$ of an orbit is found by eliminating y_1, y_2 between the two equations (19) and

$$\frac{\partial \Phi_1}{\partial y_1}\frac{\partial \Phi_2}{\partial y_2} - \frac{\partial \Phi_1}{\partial y_2}\frac{\partial \Phi_2}{\partial y_1} = 0 . \tag{21}$$

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In many problems the Hamiltonian is given in the form

$$H = \frac{1}{2}\sum_{i=1}^{n}(\omega_i^2 x_i^2 + y_i^2) + \epsilon H_\epsilon , \tag{18}$$

where ϵ is a small parameter and H_ϵ is of degree > 2. Then we find integrals of the form

$$\Phi_i = \overline{\Phi}_{i(0)} + \epsilon\, \overline{\Phi}_{i(1)} + \epsilon^2\, \overline{\Phi}_{i(2)} + \ldots, \tag{19}$$

where $\overline{\Phi}_{i(0)} = \Phi_{i2}$, and

$$\overline{\Phi}_{i,\ (\nu+1)} = -\int (\overline{\Phi}_{i,\ (\nu)}, H_\epsilon)dt . \tag{20}$$

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$$\frac{\partial \Phi_1}{\partial y_1}\frac{\partial \Phi_2}{\partial y_2} - \frac{\partial \Phi_1}{\partial y_2}\frac{\partial \Phi_2}{\partial y_1} = 0 . \tag{21}$$

In the case of a time dependent potential we take the set of the curves f when t takes all the values within a period. If these curves are closed, their outer boundary defines the space inside which the orbit is confined. These curves are in zero order parallelograms. Therefore if ϵ is sufficiently small the curves do not extend to infinity, and the orbits are confined for all times.

This is exactly true if the integrals Φ_i are convergent. In general, their convergence is unknown. However, if H is given, one can find another Hamiltonian coinciding with H up to the terms of any given degree, which has convergent integrals in a region around the origin[5].

It seems probable that even when Φ_i are not convergent, the orbits will not go to infinity if ϵ is sufficiently small.

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[5] The proof is the same as in the time independent case, G. Contopoulos, Astrophys. J. 138, 1297 (1963).

III. APPLICATION

As an example of the general case we consider the potential

$$V = \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) - \varepsilon \sin\omega t\, x_1 x_2^2 . \qquad (22)$$

Then

$$H = H_o + \varepsilon H_\varepsilon = \frac{1}{2}(y_1^2 + y_2^2 + \omega_1^2 x_1^2 + \omega_2^2 x_2^2) - \varepsilon \sin\omega t x_1 x_2^2 , \qquad (23)$$

where $y_i = dx_i/dt$.

One can find now two integrals of the form (6), namely

$$F_1 = \frac{1}{2}(\omega_1^2 x_1^2 + y_1^2) + \varepsilon\left[\frac{(y_2^2 + \omega_2^2 x_2^2)}{2\omega_2^2(\omega^2-\omega_1^2)}(\omega\cos\omega t y_1 + \omega_1^2 \sin\omega t x_1)\right.$$

$$- \frac{1}{2\omega_2^2\{(\omega^2-\omega_1^2-4\omega_2^2)^2-16\omega_1^2\omega_2^2\}}\{\omega\cos\omega t\big((\omega^2-\omega_1^2-4\omega_2^2)y_1(y_2^2-\omega_2^2x_2^2)-8\omega_1^2\omega_2^2x_1x_2y_2\big)$$

$$\left. + \sin\omega t\big(\omega_1^2(\omega^2-\omega_1^2+4\omega_2^2)x_1(y_2^2-\omega_2^2x_2^2) + 4\omega_2^2(\omega^2+\omega_1^2-4\omega_2^2)y_1x_2y_2\big)\}\right] , \qquad (24)$$

and

$$\Phi_2 = \frac{1}{2}(\omega_2^2 x_2^2 + y_2^2) - \frac{2\epsilon}{\omega_2^2\{(\omega^2-\omega_1^2-4\omega_2^2)^2-16\omega_1^2\omega_2^2\}}\Big\{ \omega\cos\omega t\Big[2\omega_2^2 y_1(y_2^2-\omega_2^2x_2^2)-\omega_2^2(\omega^2-\omega_1^2-4\omega_2^2)x_1x_2y_2\Big]$$

$$+ \omega_2^2\sin\omega t\Big[(\omega^2+\omega_1^2-4\omega_2^2)x_1(y_2^2-\omega_2^2x_2^2) + (\omega^2-\omega_1^2+4\omega_2^2)y_1x_2y_2\Big]\Big\} \quad . \qquad (25)$$

The sum of the integrals Φ_1 and Φ_2 is

$$\Phi = \Phi_1+\Phi_2 = \frac{1}{2}(y_1^2+y_2^2+\omega_1^2x_1^2+\omega_2^2x_2^2) - \epsilon\sin\omega t x_1x_2^2 + \epsilon\Bigg[\frac{(y_2^2+\omega_2^2x_2^2)(\omega\cos\omega t y_1+\omega^2\sin\omega t x_1)}{2\omega_2^2(\omega^2-\omega_1^2)}$$

$$+ \frac{1}{2\omega_2^2\{(\omega^2-\omega_1^2-4\omega_2^2)^2-16\omega_1^2\omega_2^2\}}2\Big\{-\omega\cos\omega t\Big[(\omega^2-\omega_1^2+4\omega_2^2)y_1(y_2^2-\omega_2^2x_2^2)-4\omega_2^2(\omega^2+\omega_1^2-4\omega_2^2)x_1x_2y_2\Big]$$

$$- \omega^2\sin\omega t\Big[(\omega^2-\omega_1^2-4\omega_2^2)x_1(y_2^2-\omega_2^2x_2^2) + 8\omega_2^2y_1x_2y_2\Big]\Big\}\Bigg] + \ldots \qquad (26)$$

The last integral is similar to the Hamiltonian, but there is an extra first order term in it.

A number of orbits have been calculated in this potential, with $\omega_1^2 = 0.076$, $\omega_2^2 = 0.55$, $\epsilon = 0.206$. We use these values in order to be able to compare the orbits with those calculated previously in a model of the galactic

potential near the sun[6], where

$$H = \frac{1}{2}(y_1^2+y_2^2+\omega_1^2x_1^2+\omega_2^2x_2^2) - \epsilon x_1x_2^2 - \frac{2\epsilon' x_1^3}{3} \quad . \tag{27}$$

Figures 1-4 give 4 orbits with the same initial velocity at the origin ($y_{10} = 0.0512$, $y_{20} = 0.1126$), one for the Hamiltonian (27) with $\epsilon' = 0$ and three for the Hamiltonian[23] with $\mu = 0.1$, 1 and 10.

The calculations were made for 600 time units at least with the Runge-Kutta method in double precision and a step 0.02 and/or 0.01 time units. A comparison of the results has shown that at least 7 decimal figures are always accurate.

The variations of the Hamiltonian and of the "third" integral I_2 in first approximation are given in Table I. For comparison the corresponding quantities in the case of the Hamiltonian (27) are given. There we know that inclusion of higher order terms gives a very accurate third integral[7].

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[6] G. Contopoulos, Stockholm Obs. Ann., 20, No. 6 (1958). See also ref. 2.

[7] B. Barbanis, Z. Astrophysik, 56, 56 (1962).

TABLE I.

Values of the Hamiltonian and the first order "third" integral.

		$2H_{min}$	$2H_{max}$	Φ_{init}	Φ_{min}	Φ_{max}
1	Time indep.	0.0153	0.0153	0.00131	0.00129	0.00139
2a	μ = 0.1	0.0146	0.0153	0.01492	0.01478	0.01492
2b	μ = 1	0.0132	0.0178	0.0148	0.0147	0.0161
2c	μ = 10	0.0137	0.0163	0.015299	0.015296	0.015299

The initial value of 2H is always the same (0.0153). It is seen that the "third" integral is always better conserved than the Hamiltonian.

The boundaries of the orbits in the time dependent case, are oscillating, especially in Figure 3. These boundaries can be found as follows: Equation (21) gives

$$y_1 y_2 = 0(\epsilon) \ , \tag{28}$$

hence either $y_1 = 0(\epsilon)$, or $y_2 = 0(\epsilon)$. In the first case we have in first approximation

$$\Phi_1 = \frac{1}{2}\omega_1^2 x_1^2 + \epsilon\left[\frac{2\Phi_{2;o}\omega_1^2 \sin\omega t\, x_1}{2\omega_2^2(\omega^2-\omega_1^2)}\right.$$

$$- \frac{1}{2\omega_2^2\{(\omega^2-\omega_1^2-4\omega_2^2)^2-16\omega_1^2\omega_2^2\}}\{-8\omega\,\omega_1^2\omega_2^2 \cos\omega t\, x_1x_2y_2 \qquad (29)$$

$$\left. + \omega_1^2(\omega^2-\omega_1^2+4\omega_2^2)\sin\omega t\, x_1(2\Phi_{2;o}-2\omega_2^2x_2^2)\}\right] = \Phi_{1;o}\ ,$$

where $y_2 = \pm\sqrt{2\Phi_{2;o}-\omega_2^2x_2^2}$, and $\Phi_{1;o}$, $\Phi_{2;o}$ are the values of the integrals at the initial point.

In the second case

$$\Phi_2 = \frac{1}{2}\omega_2^2x_2^2 + \frac{4\epsilon\Phi_{2;o}}{\{(\omega_1^2-\omega_1^2-4\omega_2^2)^2-16\omega_1^2\omega_2^2\}}\left[2\omega\cos\omega t\, y_1+(\omega^2+\omega_1^2-4\omega_2^2)\sin\omega t\, x_1\right] = \Phi_{2;o}, \qquad (30)$$

where $y_1 = \pm\sqrt{2\Phi_{1;o}-\omega_1^2x_1^2}$. Hence the boundaries are near the straight lines $x_1 = \pm\sqrt{2\Phi_{10}}/\omega_1$ and $y_2 = \pm\sqrt{2\Phi_{2;o}}/\omega_2$, varying periodically in time.

The deviations are larger when we are near the resonances $\omega = \omega_1$ or $\omega = \pm 2\omega_2 \pm \omega_1$, or any higher order resonance. In the above cases $\omega_1 = 0.27568$, $\omega_2 = 0.74162$, hence $2\omega_2+\omega_1 \simeq 1.76$, $2\omega_2-\omega_1 \simeq 1.21$. The last quantity is

nearest to $\omega = 1$; this is the reason why the boundaries change more in case 2b. The "third" integral is better conserved when $\omega = 10$, because this value is far from resonances. Then the space filled by the orbit is very nearly a parallelogram.

Similar results were found in many other cases. Figure 3 represents two orbits in the case $\omega_1^2 = \omega_2^2 = 0.1$, $\varepsilon = 0.1$, $\omega = 1$. The orbits are rather different from those of the corresponding time independent case (27), except for the orbit 3b, which is near a periodic orbit. In general, one expects that the resonance effects of the time-independent case, which depend on the value of ω_1/ω_2, do not affect the time dependent case, which depends on some relation of ω_1, ω_2 with ω. In the present case it happens that $2\omega_2+\omega_1 = 3x(0.3162) = 0.949$ is near $\omega = 1$, but this near resonance is of a different nature than the resonance $\omega_1 = \omega_2$. The subject is worthy of further study.

The initial conditions in the above cases are $x_{10} = y_{20} = 0$, and $y_{10} = 0.013$, $y_{20} = 0.060465$ in case 3a and $y_{10} = 0.035$, $y_{20} = 0.05099$ in case 3b. We find in case 3a: $2H_{init.} = 0.003825$, $2H_{min} = 0.00336$, $2H_{max} = 0.00417$,

$\dot{I}_{init} = 0.00319$, $\dot{I}_{min} = 0.00297$, $\dot{I}_{max} = 0.00319$, and in case 3b: $2H_{init} = 0.003825$, $2H_{min} = 0.00186$, $2H_{max} = 0.00450$, $\dot{I}_{init} = 0.00261$, $\dot{I}_{min} = 0.00240$, $\dot{I}_{max} = 0.00261$. It is seen that although we are near a resonance, the first order third integral is conserved much better than the energy.

Our experience, from the time independent cases, indicates that the conservation of the "third" integral is always improved (to a very high accuracy) as more higher order terms are included.

IV. COMPARISON WITH THE ADIABATIC INVARIANTS

The most simple adiabatic invariants are given for one-dimensional Hamiltonians that vary slowly in time. If H depends on the time through λ, where $\frac{d\lambda}{dt}$ is small with respect to $\frac{\lambda}{T}$ (T is the period of the motion), then the action integral $J = \oint y_1 dx_1$ is approximately constant[8]. Usually H is considered to vary only during a finite time interval, being constant before and after that. In a system of n degrees of freedom, n adiabatic invariants can be found when the system is separable[8].

If H is not separable but is a slowly-varying function of the time and of some variables, besides x_1, y_1, then the action $J = \oint y_1 dx_1$ is an adiabatic invariant under certain conditions that will be specified.

Gardner[9] gives a general method to construct adiabatic invariants by successive approximations. We will describe his method, and the conditions under which it is applicable.

We assume that H is a function of $t, x_1, y_1, x_2, y_2 \ldots$ and of a small parameter ϵ, that satisfies the following conditions:

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[8] L. D. Landau and E. M. Lifschitz, Mechanics, Pergamon Press, New York (1960).

[9] C. S. Gardner, Phys. Rev. 115, 791 (1959).

1) In zero order (i.e., for $\mu = 0$) it is separable.

2) It is a slowly varying function of the time and of the variables $x_2, x_3, \ldots$ (and eventually, but not necessarily of $y_2, y_3 \ldots$) i.e., it is a function of $\mu x_2, \mu x_3, \ldots$ (and, possibly, of $\mu y_2, \mu y_3, \ldots$), i.e.,

$$H = H_o(x_1, y_1, \mu x_2, y_2 \ldots \mu t) + H_o^*(y_2, \mu x_2, \ldots \mu t) \quad , \tag{31}$$

where H_o includes all terms containing x_1, y_1, and has no zero order terms in $y_2, \ldots$.

3) The curves H = const. for $\mu = 0$ and y_i = const. ($i > 1$) are closed; then they are closed also for small values of μ and fixed t, x_i, y_i ($i > 1$).

An adiabatic invariant is constructed, step by step, by successive coordinate transformations:

If we keep t and x_i, y_i ($i > 1$) constant the curves H = const. are transformed into circles by the following area preserving transformation

$$x_1' = r\cos\vartheta \quad , \qquad y_1' = r\sin\vartheta \quad , \tag{32}$$

where $r = [\, J/\pi \,]^{1/2}$ and $\vartheta = 2\pi \int^{x_1, y_1} ds/|\nabla H| \Big/ \oint ds/|\nabla H|$;

J is the area inside the curve H = const. and ds the line element along this curve. The curves H = const. are circles in the new variables.

Because of the conservation of areas $x_1' dy_1' + y_1 dx_1$ is the complete differential of a generating function F_1 and

$$x_1' = \partial F_1/\partial y_1' , \qquad y_1 = \partial F_1/\partial x_1 . \tag{33}$$

F_1 is a function of x_1, y_1', with coefficients functions of $\alpha t, \alpha x_2, y_2, \ldots$, which are kept constant in the transformation (32). As we can add any arbitrary constant in F_1 we may write

$$F_1 = x_2 y_2' + \ldots + \psi_1(x_1, y_1', \alpha x_2, y_2', \ldots \alpha t), \tag{34}$$

where ψ_1 does not contain terms independent of x_1, y_1' .

Then the equations

$$x_2' = \partial F_1/\partial y_2' = x_2 + \partial \psi_1/\partial y_2', \quad y_2 = y_2' + \alpha \partial \psi_1/\partial(\alpha x_2), \ldots , \tag{35}$$

together with equations (33) define new canonical variables x_i', y_i', with Hamiltonian

$$H' = H + \partial F_1/\partial t = H + \alpha \partial \psi_1/\partial(\alpha t), \tag{36}$$

which is of the form

$$H' = H_1(x_1'^2 + y_1'^2, \alpha t) + H_1^*(y_2', \alpha x_2', \ldots \alpha t) + \alpha \bar{H}_1(x_1', y_1', \alpha x_2', y_2', \ldots \alpha t) . \tag{37}$$

The term $\varepsilon\bar{H}_1$ contains all higher than zero order terms including x_1, y_1.

The next change of variables transforms into circles the curves $H_1 + \varepsilon\bar{H}_1 = \text{const.}$ for $t, x_i, y_i\ (i > 1)$ constant. The deviation of these curves from circles is of order ε, therefore the difference between the two sets of variables will be of order ε.

The new generating function can be written

$$F_2 = x_1' y_1'' + x_2' y_2'' + \ldots + \varepsilon\psi_2(x_1', y_1'', \varepsilon x_2', y_2'', \ldots \varepsilon t), \tag{38}$$

where ψ_2 does not contain terms independent of x_1', y_1''.

Then

$$x_1'' = x_1' + \varepsilon\partial\psi_2/\partial y_1'', \qquad y_1' = y_1'' + \varepsilon\partial\psi_2/\partial x_1'', \tag{39}$$

$$\varepsilon x_2'' = \varepsilon x_2' + \varepsilon^2\partial\psi_2/\partial y_2'', \qquad y_2' = y_2'' + \varepsilon^2\partial\psi_2/\partial(\varepsilon x_2'), \ \ldots \tag{40}$$

and the new Hamiltonian is

$$H'' = H_2(x_1''^2 + y_1''^2, \varepsilon t) + H_2^*(y_2'', \varepsilon x_2'', \ldots \varepsilon t) + \varepsilon\bar{H}_2(x_1'', y_1'', \varepsilon x_2'', y_2'', \ldots \varepsilon t). \tag{41}$$

By repeating N times this procedure we find

$$H^{(N)} = H_N\left((x_1^{(N)})^2 + (y_1^{(N)})^2, \varepsilon t\right) + H_N^*(y_2^{(N)}, \varepsilon x_2^{(N)}, \ldots \varepsilon t) + \varepsilon^N \bar{H}_N(x_1^{(N)}, y_1^{(N)}, \varepsilon x_2^{(N)}, y_2^{(N)}, \ldots \varepsilon t). \tag{42}$$

Then the quantity

$$J^{(N)} = \left((x_1^{(N)})^2 + (x_2^{(N)})^2 \right) \tag{43}$$

is constant to order N-1, i.e., $dJ^{(N)}/dt = 0(\omega^N)$. Therefore if $N \to \infty$, $J^{(N)}$ is a formal integral of motion; this is the adiabatic invariant.

If, however, both x_2 and y_2 appear in H (not through (x_2, y_2) in non linear terms, then the above method is not applicable.

E.g., in a two-dimensional system, let x_2, y_2 appear in non linear zero order terms of H_o^* and let x_2 appear also in a mixed term (including x_1 and/or y_1), of degree n in ϵ.

Then

$$y_2 = y_2' + \partial S_1/\partial x_2 ,$$

and $\partial S_1/\partial x_2$ includes terms of degree n, containing x_1 and/or y_1, because, by its definition, F_1 does not contain terms independent of x_1, y_1' except $x_2 y_2'$. Thus the expansion of H_o^* gives n-order mixed terms (containing y_2').

The next change of variables gives

$$x_2'' = x_2' + \text{n order terms containing } x_1'' \text{ and/or } y_1' + \ldots,$$

and H_1^* contains again n-order mixed terms. By n such changes of variables we can reduce all terms containing x_1 and/or y_1 up to order n-1 to a function of $(x_1^{(n)})^2 + (y_1^{(n)})^2$.

The (n+1) transformation of variables, however, cannot eliminate the n order terms that include x_1, y_1; because in the new Hamiltonian the zero order terms of H_N^* will give again mixed n order terms including $x_1^{(n+1)}$ and/or $y_1^{(n)}$.

In Gardner's paper[9] the Hamiltonian considered is of the form (3), but the above conditions are not explicitly stated.

The adiabatic invariant $J^{(N)}$ is equal to the action J if H does not depend on the time and on the variables x_2, y_2 ... Therefore if for $t < t_1$ and $t > t_2$ the variables x_i, y_i are such that H has zero derivatives of all orders with respect to t, x_2, y_2, ... , then the action $J = \oint y_1 dx_1$ is well defined for $t < t_1$ and $t > t_2$; its change during the time $t_2 - t_1$ is of order higher than any ω^N (it is at least of order $\exp(-\alpha/\omega)$, where α is a constant[10,11]).

In general, however, there is no time interval during which H = const. and no space where H is independent of x_2, y_2 ... .

In the case (27), we know that we can make the system separable by a formal variable transformation, known as the von Zeipel method. (This method gives the third integral in a somewhat different way than described above (cf. equation(14)).[5]

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[10] F. Hertweck and A. Schlüter, Z. Naturforsch. 12a, 899 (1957).

[11] P. Vandervoort, Ann. Physics, 12, 436 (1961).

If we set $\varepsilon' = 0$ we find explicitly a generating function

$$S = x_1 y_1' + x_2 y_2' + \frac{\varepsilon}{2}\left\{\frac{-y_1(y_2^2+\omega_2^2 x_2^2)}{\omega_1^2\omega_2^2} + \frac{y_1(y_2^2-\omega_2^2 x_2^2)+4\omega_2^2 x_1 x_2 y_2}{\omega_2^2(\omega_1^2-4\omega_2^2)}\right\} + \ldots \qquad (44)$$

that gives

$$y_1' = y_1 - \frac{2\varepsilon x_2 y_2}{\omega_1^2-4\omega_2^2} + \ldots \qquad (45)$$

etc. Then

$$J = 2\int_{x_1 min}^{x_1 max} y_1 dx_1 = 2\int_{x_1 min}^{x_1 max} y_1^2 dt = 2\int_{x_1 min}^{x_1 max}\left(y_1'^2 + \frac{4\varepsilon y_1' x_2' y_2'}{\omega_1^2-4\omega_2^2} + \ldots\right) dt . \qquad (46)$$

The quantity

$$2\int_{x_1 min}^{x_1 max} y_1'^2 dt$$

is a constant, with an error of order higher than the first; in fact we notice that it is exactly constant if $x_1 min$, $x_2 max$ are replaced by $x_1' min$, $x_1' max$, and that the value of y_1' for $x_1' min$, $x_1' max$ is zero. But the quantity

$$\varepsilon \int_{x_1 \min}^{x_1 \max} y_1' x_2' y_2' dt$$

is not constant, in general. Therefore J has variations of the first order in ε. Thus the adiabatic invariant $J^{(N)}(N \to \infty)$ is more general than the action $J = \oint y_1 dx_1$.

A comparison of the adiabatic invariants in the form $J^{(N)}$ with the "third" integral shows the following:

a) Both are formal series expansions in terms of small parameters (or some small parameters). When the "third" integral is given in power series in the variables and no small parameter formally appears, we may consider as small parameter the energy itself. In fact an expansion (27) may be written in dimensionless form

$$\omega_1^2 \left(\frac{x_1}{\sqrt{2H}}\right)^2 + \left(\frac{y_1}{\sqrt{2H}}\right)^2 + \omega_2^2 \left(\frac{x_2}{\sqrt{2H}}\right)^2 + \left(\frac{y_2}{\sqrt{2H}}\right)^2 - 2\varepsilon\sqrt{2H}\left(\frac{x_1}{\sqrt{2H}}\right)\left(\frac{y_2}{\sqrt{2H}}\right)^2 - \frac{2\varepsilon'\sqrt{2H}}{3}\left(\frac{x_1}{\sqrt{2H}}\right)^3 = 1, \tag{47}$$

and the small parameters are essentially $\varepsilon\sqrt{2H}$, $\varepsilon'\sqrt{2H}$, and if ε and ε' are constant the small parameter is $\sqrt{2H}$.

The difference is that the small parameter in the case of the "third" integral refers to a term, or terms, while in the case of the adiabatic invariants it refers to a variable, or variables.

b) The "third" integral is more general in the sense that it does not require H to depend on $\mu x_2 \ldots$, rather than $x_2 \ldots$ In the case of the Hamiltonian (27) we cannot find an adiabatic invariant, because both x_2 and y_2 appear in zero order. On the other hand if μ_2 is small with respect to μ_1, we may construct an adiabatic invariant expansion in powers of μ_2. This expansion is preferable, because the third integral expansion is not valid when $\mu_1 \to 0$ or $\mu_2 \to 0$.

c) The adiabatic invariants are more general in the sense that they apply also to non periodic time dependent Hamiltonians.

d) The practical construction of a "third" integral when H is a series is comparatively easy. The formulae for finding higher order terms are given, and the necessary algebra can be performed by an electronic computer . In the case of

the adiabatic invariants the changes of variables that transform the curves H_o = const. into circles cannot, in general, be given analytically in a simple form. In practice one should expand in series of another small parameter also, which is the parameter that measures the deviations of the equipotential lines (in the x_1, y_1 plane) from circles, i.e., it is essentially the parameter used in the third integral.

On the other hand if the Hamiltonian cannot be expanded in a power series, the method of the third integral may not be applicable, (except in special cases like the restricted three-body problem, etc.), while Gardner's method is in principle still valid.

We apply now both methods to the simple one-dimensional Hamiltonian

$$H = \frac{1}{2}(\omega_1^2 x_1^2 + y_1^2) - \varepsilon \sin\omega t \, x_1^2 \quad . \tag{48}$$

This case can be reduced to the well known Mathieu equation[12]

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[12] N.W. McLachlan, Theory and Application of Mathieu Functions, Clarendon Press, Oxford (1947), pp. 77,90.

$$\frac{d^2y}{dz^2} + (a-2q\sin 2z)y = 0 \ , \tag{49}$$

by setting

$$y = x_1 \ , \quad \omega t = 2z, \quad a = 4\omega_1^2/\omega^2, \quad q = 4\varepsilon/\omega^2 \ . \tag{50}$$

In this case the transformations proposed by Gardner can be explicitly carried out and we can compare the adiabatic invariant directly with the "third" integral. The "third" integral in second order approximation is

$$\begin{aligned} J = {} & \tfrac{1}{2}(\omega_1^2 x_1^2 + y_1^2) + \frac{2\varepsilon\left(\omega\cos\omega t\ x_1 y_1 - \sin\omega t\,(y_1^2 - \omega_1^2 x_1^2)\right)}{\omega^2 - 4\omega_1^2} \\ & + \frac{\varepsilon^2}{2\omega_1^2(\omega^2 - 4\omega_1^2)}\Big[\cos 2\omega t\ (y_1^2 + \omega_1^2 x_1^2) - 2(y_1^2 - \omega_1^2 x_1^2) \\ & - \frac{1}{(\omega^2 - \omega_1^2)}\Big\{(\omega^2 + 2\omega_1^2)\cos 2\omega t\,(y_1^2 - \omega_1^2 x_1^2) + 6\omega\omega_1^2 \sin 2\omega t\ x_1 y_1\Big\}\Big] + \ldots \end{aligned} \tag{51}$$

In applying Gardner's method we have to calculate the area J of the ellipse (48). Using formulae (32) we find

$$r = \frac{\sqrt{2H}}{\sqrt[4]{\omega_1^2 - 2\varepsilon\sin\omega t}} \quad , \tag{52}$$

and

$$\theta = \cos^{-1}\left(\frac{\sqrt{\omega_1^2 - 2\varepsilon\sin\omega t}}{\sqrt{2H}} x_1\right) . \tag{53}$$

Then

$$x_1' = \sqrt[4]{\omega_1^2 - 2\varepsilon\sin\omega t}\; x_1 , \qquad y_1 = \sqrt[4]{\omega_1^2 - 2\varepsilon\sin\omega t}\; y_1' , \tag{54}$$

$$F_1 = \sqrt[4]{\omega_1^2 - 2\varepsilon\sin\omega t}\; x_1 y_1' \quad , \tag{55}$$

and

$$H' = \frac{\sqrt{\omega_1^2 - 2\varepsilon\sin\omega t}}{2} (x_1'^2 + y_1'^2) - \frac{\omega\varepsilon\cos\omega t\; x_1' y_1'}{2(\omega_1^2 - 2\varepsilon\sin\omega t)} . \tag{56}$$

The next change of variables is effected in a similar way. After some operations we find

$$F_2 = x_1' y_1'' - \frac{\omega\varepsilon\cos\omega t (y_1''^2 - x_1'^2)}{8(\omega_1^2 - 2\varepsilon\sin\omega t)^{3/2}} , \tag{57}$$

$$x_1'' = x_1' - \frac{\omega\varepsilon\cos\omega t\; y_1''}{4(\omega_1^2 - 2\varepsilon\sin\omega t)^{3/2}} , \quad y_1' = y_1'' + \frac{\omega\varepsilon\cos\omega t\; x_1'}{4(\omega_1^2 - 2\varepsilon\sin\omega t)^{3/2}} , \tag{58}$$

and

$$H'' = \frac{\sqrt{\omega_1^2-2\varepsilon\sin\omega t}}{2}\,(x_1''^2+ y_1'' \quad + O(\varepsilon^2) \tag{59}$$

Similarly we find

$$x_1''' = x_1''\left(1+\frac{\omega^2\varepsilon[11\varepsilon\cos^2\omega t-4\sin\omega t(\omega_1^2 \quad n\omega t)]}{32(\omega_1^2-2\varepsilon\sin\omega t)^3}\right), \tag{60}$$

$$y_1''' = y_1'''\left(1+\frac{\omega^2\varepsilon[11\varepsilon\cos^2\omega t-4\sin\omega t(\omega_1^2-2\varepsilon\sin\omega t)]}{8(\omega_1^2-2\varepsilon\sin\omega t)^3}\right), \tag{61}$$

and

$$H = \frac{\sqrt{\omega_1^2-2\varepsilon\sin\omega t}}{2}\left(1-\frac{\omega^2\varepsilon^2\cos^2\omega t}{32(\omega_1^2-2\varepsilon\sin\omega t)^3}\right)(x_1'''^2+y_1'''^2)+O(\varepsilon^3). \tag{62}$$

The adiabatic invariant in second order approximation is

$$J^{(3)} = -(x_1'''^2+y_1'''^2) = \frac{-}{\sqrt{\omega_1^2-2\varepsilon\sin\omega t}}\Big\{y_1^2+(\omega_1^2-2\varepsilon\sin\omega t)x_1^2-\frac{\varepsilon\cos\omega t\,x_1y_1}{(\omega_1^2-2\varepsilon\sin\omega t)} \tag{63}$$

$$+\frac{\omega^2\varepsilon^2\cos^2\omega t\left(7x_1^2(\omega_1^2-2\varepsilon\sin\omega t)-5y_1^2\right)-2\sin\omega t(\omega_1^2-2\varepsilon\sin\omega t)\left(x_1^2(\omega_1^2-2\varepsilon\sin\omega t)-y_1^2\right)}{8(\omega_1^2-2\varepsilon\sin\omega t)^3}\Big\}$$

Both expansions are equivalent if ε and ω are small. If we omit all terms of order higher than two in ε or ω we find

$$\left[1 + \frac{\varepsilon^2(4\omega_1^2+3\omega^2)}{16\omega_1^2}\right]\frac{J^{(3)}\omega_1}{2\pi} = \frac{1}{2}\left\{y_1^2+\omega_1^2x_1^2 + \frac{\varepsilon\sin\omega t}{\omega_1^2}(y_1^2-\omega_1^2x_1^2)\right.$$

$$- \frac{\varepsilon\omega\cos\omega t}{\omega_1^2}x_1y_1 + \frac{\varepsilon^2\sin^2\omega t}{2\omega_1^4}(3y_1^2-\omega_1^2x_1^2) - \frac{3\varepsilon^2\omega\sin 2\omega t}{2\omega_1^4}x_1y_1 \qquad (64)$$

$$\left. + \frac{\omega^2\varepsilon\sin\omega t}{4\omega_1^4}(y_1^2-\omega_1^2x_1^2) + \frac{\omega^2\varepsilon^2}{8\omega_1^6}\left(7\omega_1^2x_1^2-5y_1^2+\sin^2\omega t(15y_1^2-13\omega_1^2x_1^2)\right)\right\}.$$

The above form of the "third" integral expansion is preferable if ε is small (q small in Mathieu's equation) if we are not near a resonance ($\omega^2 = \omega_1^2$, $\omega^2 = 4\omega_1^2$, and in general $\omega^2 = 4\omega_1^2/n^2$, i.e., $a = n^2$). The resonance cases should be treated separately.

A comparison of the values of J and $J^{(3)}$ has been made in some orbits calculated numerically, by the Runge Kutta method. In all the cases $\omega_1 = 1$, and the calculations were made for

300 time units with a step 0.02 or 0.01 time units. A check has shown that at least four significant figures in x_1, y_1 and five significant figures in H, Φ and $J^{(3)}$ are accurate. Table II tives the data and the values of the energy H, the third integral Φ (given by formula (51)) and the adiabatic invariant (given by formula (63)).

TABLE II

Comparison of the "third" integral with the adiabatic invariant.

ω_1		x_{10}	y_{10}	H_{init}	$\epsilon\sqrt{H}_{init}$	H_{max}	H_{min}	Φ_{max}	Φ_{min}	J^3_{max}	J^3_{min}
1.	0.2	0.1	0.	0.1	0.005	0.01	0.0055	0.0045	0.00499	0.00497	0.005000	0.004998
1.	0.2	0.2	0.	1.	0.5	0.2	0.38	0.59	0.50	0.48	0.500	0.499
1.	0.8	0.2	0.	1.	0.5	0.2	0.61	0.35	0.50	0.45	0.52	0.44
1.	1.2	0.2	0.	1.	0.5	0.2	0.87	0.44	0.58	0.53	0.72	0.48

It is seen that for small values of ω the second order adiabatic invariant $J^{(3)}$ is better conserved than the second order "third" integral Φ. However for ω/ω_1 approaching unity Φ is better conserved than J^3. This is more evident for larger ω. The

conservation of the zero and first order "third" integral and adiabatic invariant is always worse.

This example gives the range of values of ω for which an adiabatic invariant is useful.

If ω is near a resonance case the above formulae are no more valid, although resonances are not apparent in formula (63). In fact in a numerical example in the case $\omega_1=1$ we have found continuous increase of the amplitude of oscillations, and $J^{(3)}$ is not even approximately conserved.

This fact indicates that the action J is not an adiabatic invariant if there is a resonance between the frequency of the perturbation and the eigen-frequency of the system. This fact was known to the first authors that applied the adiabatic invariants[13], but is rarely mentioned explicitly in modern papers.

The example discussed here shows clearly the distinction between the "third" integrals and the adiabatic invariants. They are expansions in terms of different small parameters; in the case of the third integral we have a small term,

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[13] See, e.g., A. Sommerfeld, Atombau and Spektrallinien, I, 7th ed. F. Vieweg & Sohn, Braunschweig (1951) pp. 370, 698.

while in the case of the adiabatic invariants we have a slow dependence on the time and/or some variables.

This example shows further that the relative accuracy of the two expansions depends on the values of the parameters used. It indicates also the disadvantage of the adiabatic invariants, in that they cannot be used in resonance or near resonance cases.

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CAPTIONS FOR FIGURES

Fig. 1. Orbit in the potential $V = \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) - \varepsilon x_1 x_2^2$ for $\omega_1^2 = 0.076$, $\omega_2^2 = 0.55$, $\varepsilon = 0.206$, and initial conditions $x_{10} = x_{20} = 0$, $y_{10} = 0.0512$, $y_{20} = 0.1126$.

Fig. 2a,b,c Orbits in the potential $V = \frac{1}{2}(\omega_1^2 x_1^2 + \omega_2^2 x_2^2) - \varepsilon \sin \omega t\, x_1 x_2^2$ for the same constants and initial conditions and (a) $\omega = 0.1$, (b) $\omega = 1$, (c) $\omega = 10$.

Fig. 3a,b Orbits in the same potential as in Fig. 2, for $\omega_1^2 = \omega_2^2 = \varepsilon = 0.1$, $\omega = 1$ and initial conditions $x_{10} = x_{20} = 0$ and (a) $y_{10} = 0.013$, $y_{20} = 0.060465$; (b) $y_{10} = 0.035$, $y_{20} = 0.05099$.

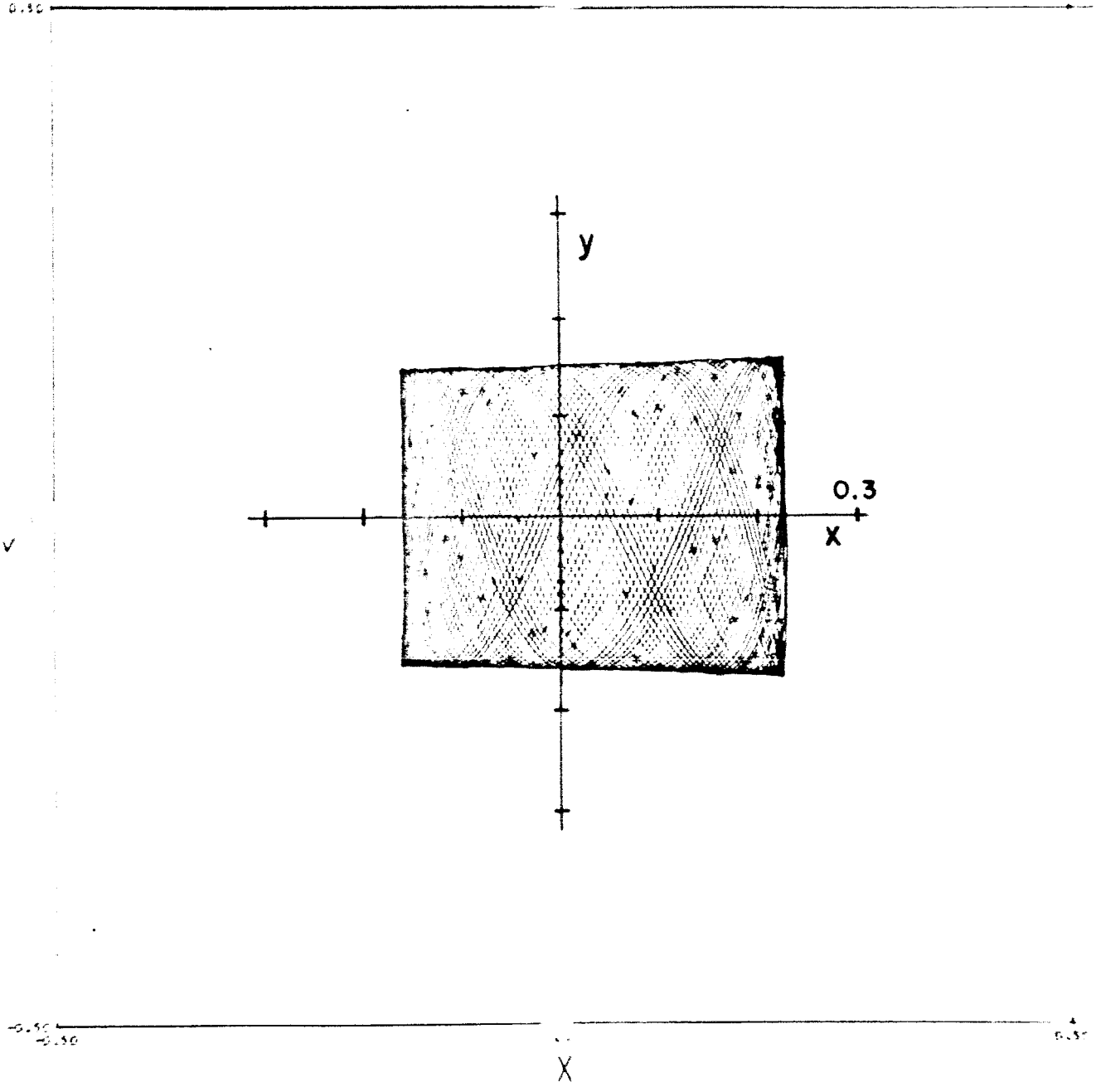
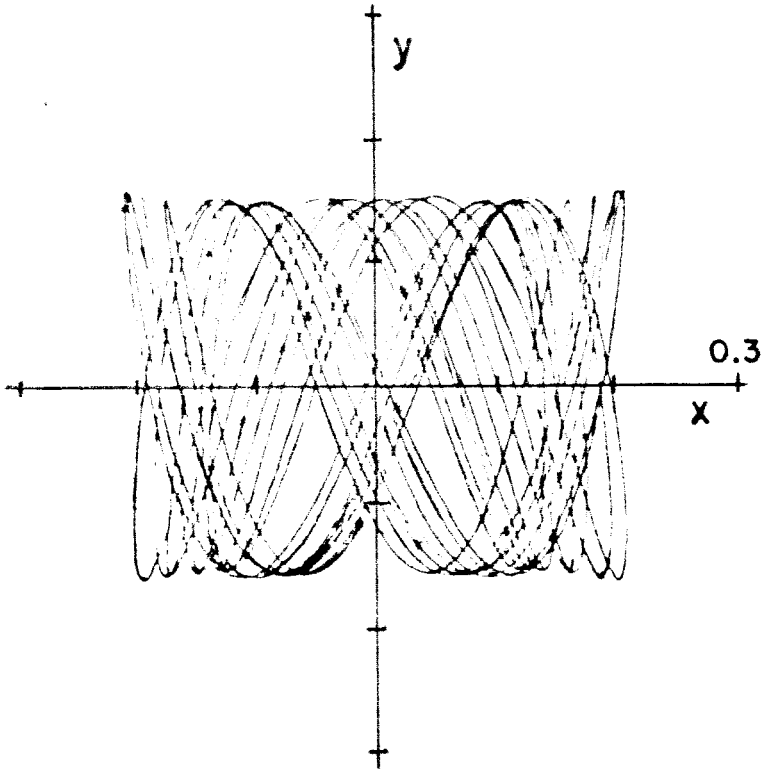


Fig 1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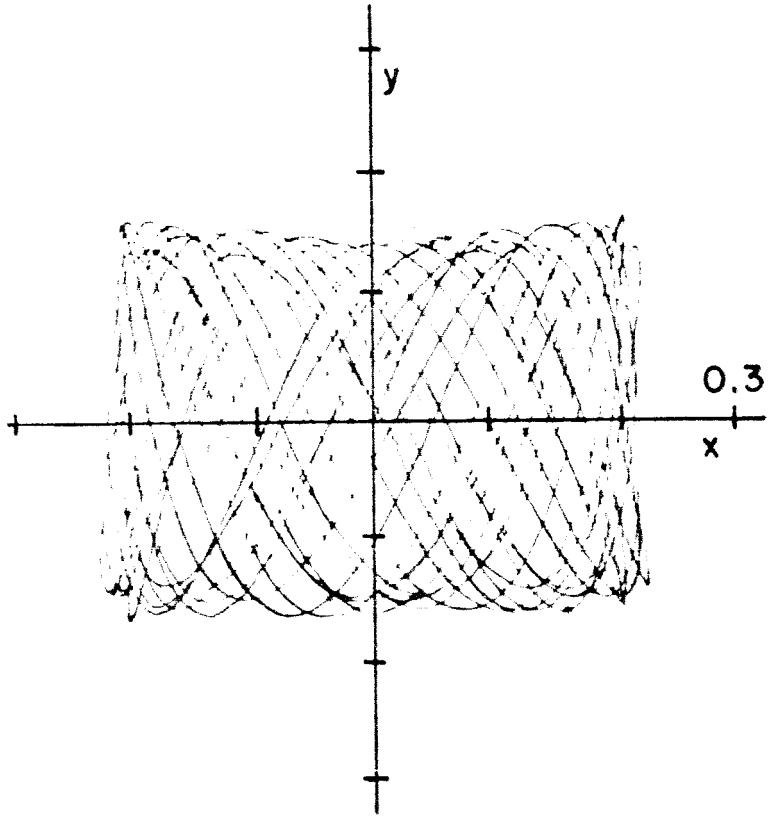
x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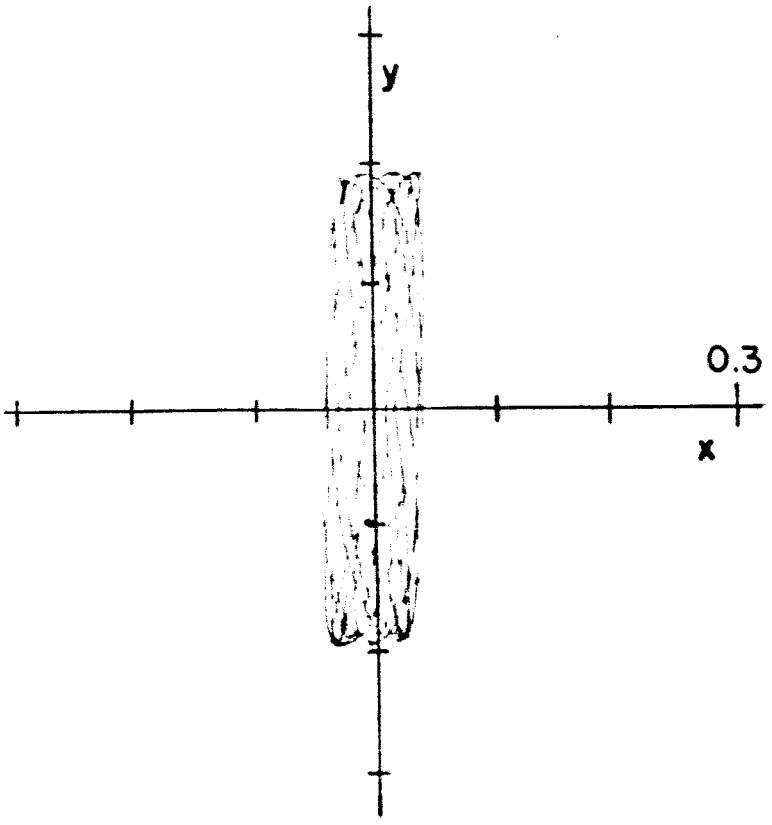
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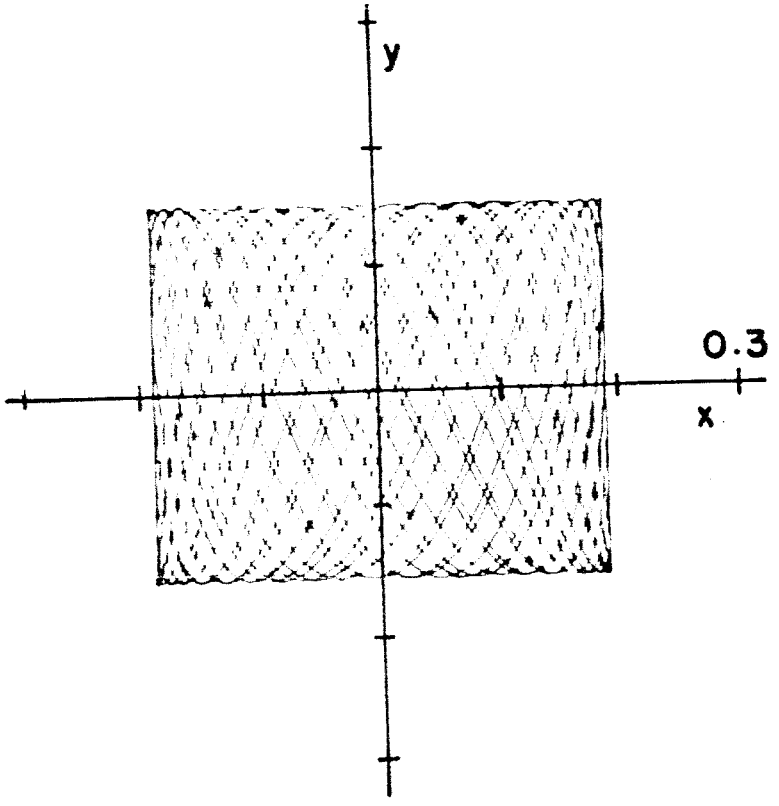
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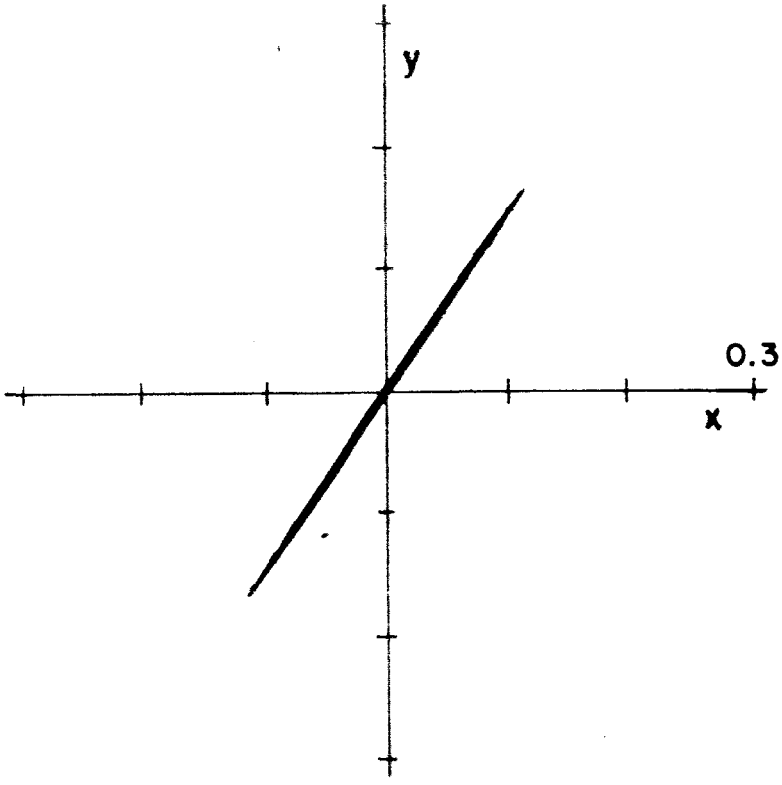

Fig. 5